Extended W-twisted order:

For $\mu, \nu \in X_*$, $\mu \geq \nu \iff \mu - \nu \geq 0 \iff \mu - \nu = \sum_{i \in I} n_i \alpha_i^\vee \quad n_i \geq 0$

$\iff \langle \varpi_i, \mu - \nu \rangle = n_i \in \mathbb{Z}_{\geq 0} \ \forall i$

Extend on $t_{\mathbb{R}}$ by declaring $\alpha \geq \beta$ if $\langle \varpi_i, \alpha - \beta \rangle \geq 0 \quad \forall i \in I$.

W-twist: $\mu \geq_w \nu \iff w^{-1}\mu \geq w^{-1}\nu \iff \langle \varpi_i, w^{-1}(\mu - \nu) \rangle \geq 0 \ \forall i \iff \langle w\varpi_i, \mu - \nu \rangle \geq 0 \ \forall i$

Notation:

$\mu \in X_*(T)$, $w \in W$

$C_w^\mu := \{\alpha \in t_{\mathbb{R}} : \alpha \geq_w \mu\} = \{\alpha \in t_{\mathbb{R}} ; \langle w\varpi_i, \alpha \rangle \geq \langle w\varpi_i, \mu \rangle \ \forall i \in I\}$

vertex of the cone is $\mu_w$, normals are $w\varpi_i$

$\Gamma = \{ w\varpi_i ; i \in I, w \in W\}$ = chamber weights

Recall. We defined a family of polytopes living in $t_{\mathbb{R}}$ called GGMS polytopes (or pseudo-Weyl polytopes). They can be defined in two ways.

1. From a collection of coweights (which turn out to be the vertices of P)

   This defines P in terms of intersections of translated reflected cones.

2. From a collection of integers (which turn out to define the faces of P)

   This defines P as intersections of half-spaces.

## §.1. Polytopes from a collection of coweights

Given a collection of coweights $(\mu_w)_{w \in W}$, define

$$P(\mu_\bullet) = \cap_{w \in W} C_w^{\mu_w}$$

If $\mu_v \geq_w \mu_w \ \forall w \in W$, then $\mu_v \in C_w^{\mu_w} \ \forall w \in W \Rightarrow \mu_v \in P(\mu_\bullet) \ \forall v \in W$.

We impose that $\{\mu_w\}_{w \in W}$ satisfies $\mu_v \geq_w \mu_w \ \forall v, w \in W$. Then:

Prop 2.2. i) The vertices of $P(\mu_\bullet)$ are the $\{\mu_w ; w \in W\}$ (repetition is allowed). Thus, $P(\mu_\bullet) = \mathrm{conv}(\mu_\bullet)$.

Given $\lambda \in (X_*)_+$, we know that the irrep $V(\lambda)$ of $G^\vee$ is such that

$$wt(V(\lambda)) = \{\mu \in (X_*)^+ ; V(\lambda)_\mu \neq 0\} = \mathrm{conv}(W\lambda).$$

This polytope is called the $\lambda$-Weyl polytope. Moreover, if $\mu \in wt(V(\lambda))$, then $\mu \leq \lambda$ and ($\mu \geq \underset{\text{lowest weight of } V(\lambda)}{w_0\lambda}$.)

One can declare $\mu_w := w w_0 \lambda$ and it follows that $\mu_v \geq_w \mu_w \ \forall v, w \in W$. This is true since $\forall w, v \in W$,

$(w^{-1}v w_0)\lambda \geq w_0\lambda \iff \forall i \in I, \langle \varpi_i, w^{-1}v w_0 \lambda \rangle \geq \langle \varpi_i, w_0\lambda \rangle$

$\iff \forall i \in I, \langle w\varpi_i, \underbrace{v w_0 \lambda}_{\mu_v} \rangle \geq \langle w\varpi_i, \underbrace{w w_0}_{\mu_w} \lambda \rangle$

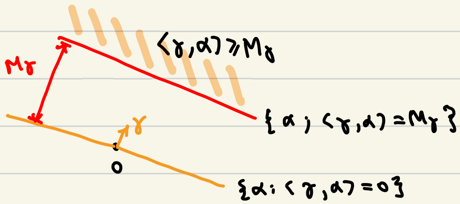
$\iff \mu_v \geq_w \mu_w.$

## §.2 Polytopes from collections of integers

Given a collection of integers $\{M_\gamma\}_{\gamma\in\Gamma}$ define

$$P(M_\bullet) = \bigcap_{\gamma\in\Gamma} \{\alpha \in \mathfrak{t}_{\mathbb{R}} ; \langle\gamma,\alpha\rangle \geq M_\gamma\}$$

intersection of half spaces




Let $\{\mu_w\}_{w\in W}$ be a collection of coweights s.t. $\mu_v \geq_w \mu_w$. Then, letting $M_{w\varpi_i} = \langle w\varpi_i, \mu_w\rangle$, one has automatically $P(M_\bullet) = P(\mu_\bullet)$.

In the case of $\lambda$-Weyl polytopes, $M_{w\varpi_i} = \langle w\varpi_i, ww_0\lambda\rangle = \langle\varpi_i, w_0\lambda\rangle$ ← independent of $w$!

$= \langle w_0\varpi_i, \lambda\rangle$ ← since $w_0^{-1} = w_0$

$= \langle -\varpi_{i^*}, \lambda\rangle = -\langle\varpi_{i^*}, \lambda\rangle$

Dynkin involution

The question is now when is a collection $\{M_\gamma\}_{\gamma\in\Gamma}$ coming from a collection $\{\mu_w\}_{w\in W}$? Notice that when $M_\gamma$ comes from a collection of coweights, then

$$M_{ws_i\varpi_i} + M_{w\varpi_i} + \sum_{j\neq i} a_{ji} M_{w\varpi_j} \leq 0 \quad \star$$

since

$$\underbrace{\langle ws_i\varpi_i, \mu_{ws_i}\rangle}_{M_{ws_i\cdot\varpi_i}} + \underbrace{\langle w\varpi_i, \mu_w\rangle}_{M_{w\varpi_i}} + \sum_{j\neq i} a_{ji} \underbrace{\langle w\varpi_j, \mu_w\rangle}_{M_{w\varpi_j}}$$

$\mu_v \geq_w \mu_w \iff \langle w\varpi_i, \mu_w\rangle \leq \langle w\varpi_i, \mu_v\rangle$

$\mu_w \geq_{ws_i} \mu_{ws_i} \iff \langle ws_i\varpi_i, \mu_{ws_i}\rangle \leq \langle ws_i\varpi_i, \mu_w\rangle$

$$\leq \langle ws_i\varpi_i, \mu_w\rangle + \langle w\varpi_i, \mu_w\rangle + \sum_{j\neq i} a_{ji} \langle w\varpi_j, \mu_w\rangle$$

$$= \langle ws_i\varpi_i + w\varpi_i + \sum_{j\neq i} a_{ji}\, w\varpi_j, \mu_w\rangle$$

$$= \langle w(\underbrace{s_i\varpi_i}_{\varpi_i - \alpha_i} + \varpi_i + \textstyle\sum_{j\neq i} a_{ji}\varpi_j), \mu_w\rangle = \langle w(2\varpi_i + \textstyle\sum_{j\neq i} a_{ji}\varpi_j), \mu_w\rangle = 0$$

Call the inequalities (*) the "edge inequalities"

Prop 2.2.2) Suppose $\{M_\gamma\}_{\gamma\in\Gamma}$ satifies the edge inequalities. Then, $P(M_\bullet)$ is a GGMS polytope with vertices given by $\mu_w = \sum_{i\in I} M_{w\varpi_i} \cdot w\alpha_i^\vee$.

$$\mu_w = \sum_{j\in I} M_{w\varpi_j}\, w\alpha_j^\vee \qquad \mu_{ws_i} = \sum_{j\in I} M_{ws_i\varpi_j}\,(ws_i)\alpha_j^\vee$$

$$\mu_{ws_i} = \sum_{j\in I} M_{ws_i\varpi_j}\, ws_i\alpha_j^\vee$$

$$= \Big(\sum_{j\neq i} M_{w\varpi_j}\, ws_i\alpha_j^\vee\Big) + M_{ws_i\varpi_i}\, ws_i\alpha_i^\vee$$

$$= \Big(\sum_{j\neq i} M_{w\varpi_j}\cdot w(\alpha_j^\vee - a_{ji}\alpha_i^\vee)\Big) - M_{ws_i\varpi_i}\, w\alpha_i^\vee$$

$$= \Big(\sum_{j\neq i} M_{w\varpi_j}\cdot w\alpha_j^\vee\Big) + \Big(-M_{ws_i\varpi_i} - \sum_{j\neq i} a_{ji} M_{w\varpi_j}\Big)\, w\alpha_i^\vee$$

and

$$\mu_w = \sum_{j\neq i} M_{w\varpi_j}\cdot w\alpha_j^\vee + M_{w\varpi_i}\, w\alpha_i^\vee$$

$$\Rightarrow \mu_{ws_i} - \mu_w = \Big(-M_{w\varpi_i} - M_{ws_i\varpi_i} - \textstyle\sum_{j\neq i} a_{ji} M_{w\varpi_j}\Big)\, w\alpha_i^\vee$$

which justified the terminology "edge inequalities".

## §3. Geometric data

As mentioned during the previous weeks, there is a geometric analogue of $P(\mu_\bullet)$. Given $\{\mu_w\}_{w\in W}$ satisfying $\mu_v \geq_w \mu_w$, define

$$A(\mu_\bullet) = \cap_{w\in W} S_w^{\mu_w} = \text{GGMS stratum of } \mu.$$

Recall that $\Phi(\overline{S_w^{\mu}}) = C_w^{\mu}$

**lemma**: $\Phi(\overline{A(\mu_\bullet)}) = P(\mu_\bullet)$ or $A(\mu_\bullet) = \emptyset$

We want a geometric construction of the $P(M_\bullet)$ such that $A(M_\bullet) = A(\mu_\bullet)$.

First, we generalize the valuation on $GL_n(\mathbb{K})$ defined as $\mathrm{val}(\det(g))$.

Recall that $\nu : \mathbb{K} \to \mathbb{Z}\cup\{\infty\}$, $\nu(f) = $ degree of the first non-zero coeff. If $\nu(f) = k$, then $f = a_k t^k + \dots$ and

$$f \in t^k \mathcal{O}, \quad f \notin t^{k+1}\mathcal{O} \qquad \dots \subseteq t^k\mathcal{O} \subseteq t^{k-1}\mathcal{O} \subseteq \dots$$

Thus, one can redefine $\nu$ using the filtration $\{t^k\mathcal{O} ; k\in\mathbb{Z}\}$ of $\mathbb{K}$ by

$$\nu(f) = k \quad \text{if} \quad f\in t^k\mathcal{O} \text{ but } f\notin t^{k+1}\mathcal{O}.$$

Now, for any $\mathbb{C}$-vect space $U$, the $\mathbb{K}$-vect space $\mathbb{K}\otimes_{\mathbb{C}} U$ has a filtration by $t^k\mathcal{O}\otimes_{\mathbb{C}} U$. Define

$$\nu(u) = k \text{ if } u\in t^k\mathcal{O}\otimes U \text{ but } u\notin t^{k+1}\mathcal{O}\otimes U.$$

We use the convention $Gr = G(\mathcal{O})\backslash G(\mathbb{K})$. For each $i\in I$, fix $v_{\varpi_i}\in V(\varpi_i)$ a hw vector. For each $\gamma = w\varpi_i \in \Gamma$, let $v_\gamma = \overline{w} v_{\varpi_i} \in V(\varpi_i)$, where $\overline{s}_i = \psi_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)$.

If $V$ is a $G$ representation, $\rho : G \to GL(V) \Rightarrow \rho : G(\mathbb{K}) \to GL(V)(\mathbb{K}) = GL(\mathbb{K}\otimes V)$. This preserves the notion of hw vectors : $v\in V^N \Rightarrow v\in \mathbb{K}\otimes V^{N(\mathbb{K})}$. Also, if $v\in V_\nu$, then $t^\mu v = t^{\langle \nu,\mu\rangle} v$. lastly, if $g\in G(\mathcal{O})$, then $\rho(g)\in GL(\mathcal{O}\otimes V) \Rightarrow \det(\rho(g)) \in \mathcal{O}^\times$.

$\mathbb{K}$ linear

$G$ acts on $V(\varpi_i) \Rightarrow G(\mathbb{K})$ acts on $\mathbb{K}\otimes V(\varpi_i)$.

Thus, if $g\in G(\mathcal{O})$, $\mathrm{val}(gv) = \mathrm{val}(v)$. It descends to a function

$$D_\gamma : Gr \to \mathbb{Z}, \quad D_\gamma(g) = \mathrm{val}(g v_\gamma).$$

$$D_\gamma(gh) = \mathrm{val}(gh v_\gamma) = \mathrm{val}(h v_\gamma) = D_\gamma(h) \qquad \text{if } g\in G(\mathcal{O}).$$

Moreover, $v_{\varpi_i}$ is a hw vector $\Rightarrow$ it is $N$ invariant $\Rightarrow$ $v_{\varpi_i} \in \mathbb{K} \otimes V(\varpi_i)$ is $N(\mathbb{K})$ invariant. Using the $W$-twist, we see that $\bar{w} v_{\varpi_i}$ is $\bar{w} N(\mathbb{K}) \bar{w}^{-1} = N_w(\mathbb{K})$-invariant.

Recall that

$$S_w^\mu = t^\mu N_w(\mathbb{K}) \subseteq G(\mathcal{O}) \backslash G(\mathbb{K}) \quad , \quad Gr = \bigsqcup_{\mu \in X_*} S_w^\mu \qquad (**)$$

Also, if $\bar{w} \in N(T)$ is any lift of $w \in W$, then $\bar{w} v \in V_{wv}$.

Consequently,

$$t^\mu \cdot v_\gamma = t^{\langle w\varpi_i, \mu \rangle} \cdot v_\gamma$$

and $x \in S_w^\mu \Rightarrow x = t^\mu \bar{w} n \bar{w}^{-1} \quad n \in N(\mathbb{K})$,

$$D_\gamma(x) = \mathrm{val}(t^\mu \bar{w} n \bar{w}^{-1} \bar{w} v_{\varpi_i}) = \mathrm{val}(t^\mu \cdot \bar{w} v_{\varpi_i})$$

$$= \mathrm{val}((t^{\langle w\varpi_i, \mu \rangle}) \bar{w} v_{\varpi_i}) = \langle w\varpi_i, \mu \rangle.$$

It follows that $D_\gamma$ takes constant values on $S_w^\mu$. Since for a fixed $w$, $\langle w\varpi_i, - \rangle$ $\forall i \in I$ determines $\mu$, it follows from the decomp $**$ that

$$S_w^\mu = \{ L \in Gr \; ; \; D_{w\varpi_i}(L) = \langle w\varpi_i, \mu \rangle \;\; \forall i \in I \}.$$

This prompts the following def: For $\{M_\gamma\}_{\gamma \in \Gamma}$ a collection of integers

$$A(M_\bullet) = \{ L \in Gr \; ; \; D_\gamma(L) = M_\gamma \;\; \forall \gamma \in \Gamma \}$$

and it follows that $A(M_\bullet) = A(\mu_\bullet)$.

Upshot: There is a condition of $\{M_\bullet\}$ s.t. $\Phi(\overline{A(M_\bullet)}) \neq \emptyset$ iff this condition is satisfied.